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THE EFFECT OF LOADING PATH ON THE YIELD SURFACE AT ELEVATED TEMPERATURES

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Abstract—The results of an experimental study of yield surfaces of pure aluminum at elevated temperatures from 70 to 305°F are presented. Several prestressing paths in σ–τ space are used. On the basis of these results a new law of hardening is formulated. In addition it is shown that the yield surfaces do not pass through the prestress points and that even at a relatively small prestress the yield surfaces do not enclose the origin.

INTRODUCTION

THIS paper presents the results of experiments in which thin-walled tubes of commercially pure aluminum in the annealed condition were loaded in combined tension and torsion in the plastic region at elevated temperatures. The purpose of these tests was to determine the yield surface of pure aluminum in the tension–torsion space as a function of temperature for several prestressing paths. This paper presents an extension of previous work by the senior author and his associates [1, 2]. While in the previous work the prestressing was only torsional, in the present paper a variety of prestressing paths are explored. As in [1, 2] so here also the temperature range investigated was $70°\text{F} \leq T \leq 305°\text{F}$.

In the experimental determination of yield surfaces the definition of yielding used can greatly affect the results obtained. Onset of yielding has been variously defined as (a) the stress at the proportional limit, (b) the stress at a strain offset of a given amount and finally (c) the stress obtained by a backward extrapolation from the stress–strain curve to the elastic line. It is generally believed that the determination of the proportional limit depends on the accuracy of the instrumentation and on some degree of subjective judgment of the observer. The criticism of the offset definition of yielding is that it is arbitrary and has no deeper physical meaning. The backward extrapolation definition also has no precise physical meaning: it is simply a gross indication of yielding.

In the experimental determination of the yield surfaces it is of importance whether one, or more than one, specimen is used for obtaining the complete yield surface and the subsequent yield surfaces. If more than one specimen is used, then there exists an inevitable scattering of the results. Since we wish to avoid scattering of the results we shall use only one specimen. However, if only one specimen is used, then in order to obtain each indication of yield, it is necessary to probe into the plastic region and therefore to deform the yield surface while trying to determine it. Accordingly, only when the proportional limit definition of yielding is used and the incursion into the plastic region is extremely small—

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of the order of a few microinches per inch—will it be possible to use only one specimen for the entire determination with negligible distortion of the yield surface and of the subsequent yield surfaces.

In the present experiments we proceeded as in the experiments reported in [1, 2]; that is, only one specimen was used for the determination of the entire virgin yield surface and its subsequent yield surfaces. Each incursion into the plastic region is limited to approximately 3 μin./in.; the sensitivity of the instrumentation is of the order of $\frac{1}{2}$ μin./in. The procedure for detecting the proportional limit consists of defining it on the stress–strain diagram as corresponding to the intersection of two straight lines, the elastic line and a strain-hardening line; the latter is the line passing through the first three consecutive points deviating to the same side of the linear elastic line. This procedure bears a superficial resemblance to the backward-extrapolation technique, but is significantly different. The present procedure utilizes the first strain-hardening line indicated after elastic response, whereas the backward-extrapolation technique relies on establishing an "asymptotic" strain-hardening line; thus the present procedure is sensitive to the very earliest positive indication of yielding with the result that the plastic incursions are minimized. By introducing this operational definition of the proportional limit we succeeded in receiving a striking consistency in the experimental results.

The experiments were made using five thin-walled tubes of commercially pure aluminum in the annealed condition, loaded in combined tension and torsion in the temperature range 70–305°F. Each specimen was used to obtain a virgin yield surface and a series of subsequent yield surfaces; each of the subsequent yield surfaces is associated with one of several levels and directions of prestressing at 70°F. The yield surfaces were determined in tension–torsion temperature space by smoothly connecting the four yield curves obtained at 70, 151, 227 and 305°F.

It is shown that the virgin yield surface is an elliptical cone truncated between 70 and 305°F, with the base at 70°F being nearly the Mises ellipse.

Restricting ourselves to yield curves at room temperature, we observe that increasing prestressing in any direction from one stress level to another, even while changing the direction of prestressing, moves and distorts the yield curve in a manner which qualitatively, and to some extent quantitatively, is predictable. In particular, suppose that the yield curve at some level of prestressing is C_i and that the stress path which leads to the yield curve, C_i, terminates at the stress point, P_i, Fig. 1. As shown in the section on test procedure, the stress point retraces the stress path backwards and finally reaches some position, 0, inside C_i, after verifying the yield curve, C_i. Suppose now that an additional prestressing is generated by a motion of the stress point from inside C_i to a position, P_{i+1}, outside C_i. Then the present experiments show that the new yield curve, C_{i+1}, corresponding to the stress path terminating at P_{i+1}, is generated from the yield curve, C_i, by a superposition of a rigid body translation in the direction of prestressing and of a deformation in the same direction. The rigid body translation is illustrated in Fig. 1 by the motion of curve C_i to C'_{i+1}. The deformation is illustrated by the change from curve C'_{i+1} to C_{i+1}. The effect of the deformation is that the width of the yield curve in the direction of prestressing will usually decrease. In particular the motion of the forward part, *ABD*, of the yield curve will usually be less than the motion of its rear part, *AED*. The amount of rigid body translation is determined by the motion of the straight line, *AD*, to the new position, *KL*.

The motions of the yield curves at higher temperatures but due to prestressing at 70°F follow the same behavior as illustrated above, as will be seen later in this paper.

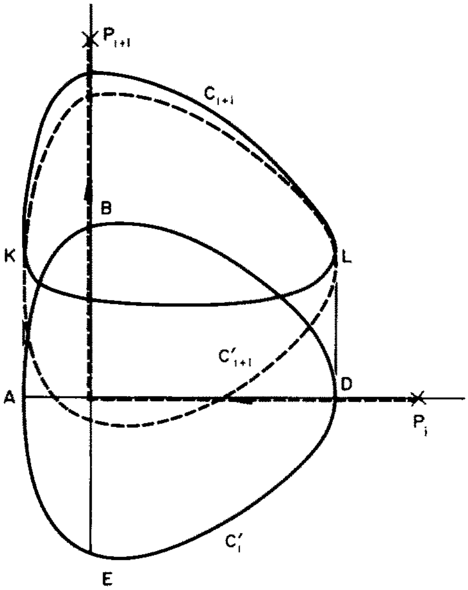

FIG. 1. Hardening law.

Furthermore, the yield surfaces do not pass through the prestress points and even at a relatively small prestress the yield surface does not enclose the origin.

In order to place the work presented in this paper in its proper historical perspective it is worthwhile to mention that the determination of yield curves at high temperature has not been attempted before except by the senior author and his associates in [1, 2]. On the other hand the determination of yield curves at room temperature has been the subject of numerous investigations as, for example, in the papers by Naghdi *et al.* [3], Ivey [4], Batdorf and Budiansky [5], Bertsch and Findley [6], Taylor and Quinney [7], Hu and Bratt [8], Miastkowski and Szczepinski [9], Mair and Pugh [10], Dudderar and Duffy [11] and others.

We believe that the considerable improvement in the consistency of the results presented in this paper as compared with previous investigations, a consistency which allows us to formulate unambiguously the hardening law presented above, is due to a significant extent to the new definition of yielding introduced in Refs. [1, 2].

The hardening law introduced in this paper is quite different from hardening laws introduced previously by other authors on the basis of theoretical or practical considerations as, for example, by Prager [12], Ishlinskii [13], Hodge [14] and Ziegler [15]. The present hardening law has been obtained entirely from experimental observations and thus it may offer a more accurate representation of the real behavior of aluminum than the hardening laws proposed previously.

EXPERIMENTAL DETAILS

The specimens used in these experiments were made from commercially pure aluminum 1100-0. The final nominal inside diameter, wall thickness and length of the uniform section of each specimen were $2\frac{1}{16}$, 0·050 and 4 in., respectively. The maximum variations of the wall thickness and the outside diameter from the average value were 0·00045 and 0·00020 in., respectively. The specimens were annealed at 650 ± 10°F for 1 hr and allowed to furnace cool for 24 hr.

Two 45° rosette BLH-FABR-50-12S13 type strain gages were bonded to the outer surface of the specimen at midlength in locations diametrically opposite to each other. These gages permit long-term strain measurements at temperatures up to 450°F and are factory compensated for minimum temperature effect. The rosette orientation was such that one gage element was in alignment with the longitudinal axis of the specimen and the other two elements were in alignment with the 45° angles to each side of the longitudinal axis. The main advantage of this type of arrangement is that, regardless of the loading path in stress space, at least two independent strain readings were simultaneously indicated. This procedure guarantees that at least two stress–strain curves can be used simultaneously for the determination of the yield point which will give a higher accuracy than if only one stress–strain curve would have been available.

The Wheatstone bridge circuit used in connecting the strain gages to the strain indicators was arranged in such a way that the active gages were mounted on opposite arms. Consequently, it was possible to read the *apparent strain* which is twice the actual strain; that is, the output signal is twice as large as that obtained with only one active gage, with the same equipment.

Three BLH model 800 series 80360 strain indicators with digital readout were used. Their readability is one microinch/inch apparent strain which is equal to $\frac{1}{2}$ μin./in. actual strain.

The testing machine used was of dead weight type allowing the rate of loading to be controlled. This machine can be used to apply axial load, torsion and reverse torsion, at variable temperature. A complete description of this testing machine is given in Ref. [2].

The thermocouples used in these experiments were of the beaded junction type, Omega 0·010 in. diameter, alumel–chromel wire. They were connected to a Honeywell Electronik 15 strip chart recorder through a thermocouple selector switch to measure the temperatures of the specimen at selected points at midlength of the specimen where the strain gages were located.

The temperature of the specimen was increased by heat conduction. Heat was supplied by two 250 W heating coils, one attached at each end of the specimen around the circumference. A continuously variable transformer was used for regulating the line voltage of the specimen heating coils and therefore the temperature.

In the present tests in order to improve on the accuracy the usual means of dummy gage compensation was not employed but the dummy specimen was maintained at the constant temperature of 95°F. The reason for the selection of this particular temperature is that it was observed experimentally that in the region 95 ± 10°F the temperature–strain curve for the temperature compensated gages used has a zero slope. The reasoning on which the above technique is based is given in detail in Ref. [2].

DETERMINATION OF THE YIELD POINT

In these experiments the following procedure was used in determining the yield point (see also Ref. [2] where more details are given).

In torsion each standard loading step was two pounds (110 psi of shearing stress). In tension the load increment was 50 lb (150 psi tensile stress). The apparent strain increments were 27–32 μin./in. for both loadings within the temperature range 70–305°F.

Each strain increment obtained from a constant step loading was plotted on a load–strain diagram. The readings were plotted as short horizontal lines 1 μin./in. wide and centered about the observed values.

To reduce the horizontal spread of the graph, the strain increment less an arbitrary constant was plotted on the diagram. Three load–strain diagrams were obtained simultaneously from the continuous loading, one from each element of the rosette.

After a small number of step loadings, a straight line—elastic line—could be traced through the strain readings for an elastic region. If more than two consecutive points went off the elastic line at the same stress level for all three diagrams, then the material was considered yielded. One more step loading was added for further confirmation: in no test did this procedure result in an accumulation of more than 5 μin./in. apparent plastic strain.

The final test load was placed on the machine for a minimum of 2 min. Unloading followed and the strain increments drawn on the same diagram along the side of the loading path. An elastic line was again determined for the unloading path in a manner similar to that described above. The loading and unloading lines were verified to be parallel and since they were not coinciding it was clear that the material had yielded.

After the elastic lines were determined on the three load–strain curves described above, straight lines were drawn to connect those strain readings immediately deviating from the elastic lines. The intersection of the elastic line and the short straight line over the small plastic region was defined as the yield point. This method usually gave a unique yield point from the three load–strain diagrams. If the three diagrams did not give the same yield point, average values were used.

TEST PROCEDURE

To make clear the procedure for determining the yield surface, the loading program for a typical specimen, *S*-5, is described in detail. The investigation started with the virgin specimen at 70°F temperature. The specimen was installed on the testing machine and because of the initial dead weight of the torsion mechanism suspended from the specimen, the initial stress state was $\tau = 0$, $\sigma = 50$ psi, which is referred to as the *minimal stress state* and is denoted as point 1 in Fig. 2.

The stress paths along which the probing of each yield surface was made are illustrated in Fig. 2. From point 1 loading in torsion proceeded to point 2 where yielding occurred; then the stress path included unloading to point 1 and loading to torsion to point 3 where yielding occurred again. The paths follow the straight line segments connecting the points denoted by increasing numbers until final unloading occurs at point 14, which coincides with point 1.

Upon completion of the investigation on the 70°F isothermal plane, the temperature of the specimen was raised to 151°F and maintained at that level while the same stress path described above was applied. The procedure was repeated at 227 and 305°F, successively, and thus the four yield curves in Fig. 2 were obtained. These curves represent the intersections of the initial yield surface with the four isothermal planes.

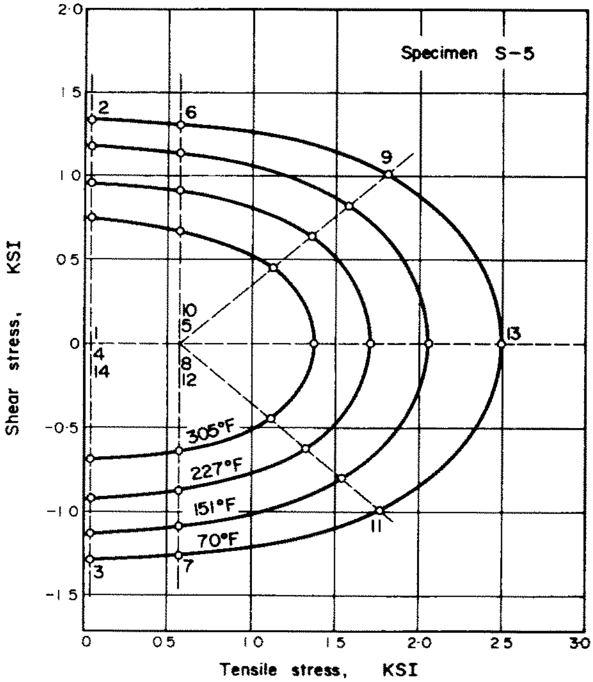

FIG. 2. Initial yield surface for specimen *S*-5.

The specimen was allowed to cool to 70°F and then prestressed in tension to a level denoted by *A* in Fig. 3. To obtain the first subsequent yield surface the loading path 1–29 shown in Fig. 3 was followed where the prestress level *A* is point 2. It is seen that at the 70°F temperature the specimen was first partly unloaded in tension until yielding occurred at the point 3 and then it was reloaded in tension until yielding occurred again at the point 4. Then the load was reduced to a position *Z* slightly below midway between the points 3 and 4. As with the initial yield surface, the first subsequent yield surface was determined by generating the isothermal yield curves at 70, 151 and 305°F, successively.

This gave the four yield curves shown in Fig. 3. The specimen was then brought back to 70°F and prestressed in tension to a still higher level. The previous procedure was repeated in order to obtain the second subsequent yield surface shown in Fig. 4. Then the previous procedure was repeated twice again to obtain the third and fourth subsequent yield surfaces shown in Fig. 4. For the third and fourth subsequent yield surfaces the loading paths are slightly different than before, and appropriate to the form of the yield surface.

The rate of loading was 3 min for each loading increment during prestressing or unloading and 1 min for each loading increment while probing within a yield surface. The rate of temperature change during heating was 3°F/min. The total time for prestressing from one level to the next was approximately 4 hr while one day minimum was required to obtain each yield curve at a given temperature.

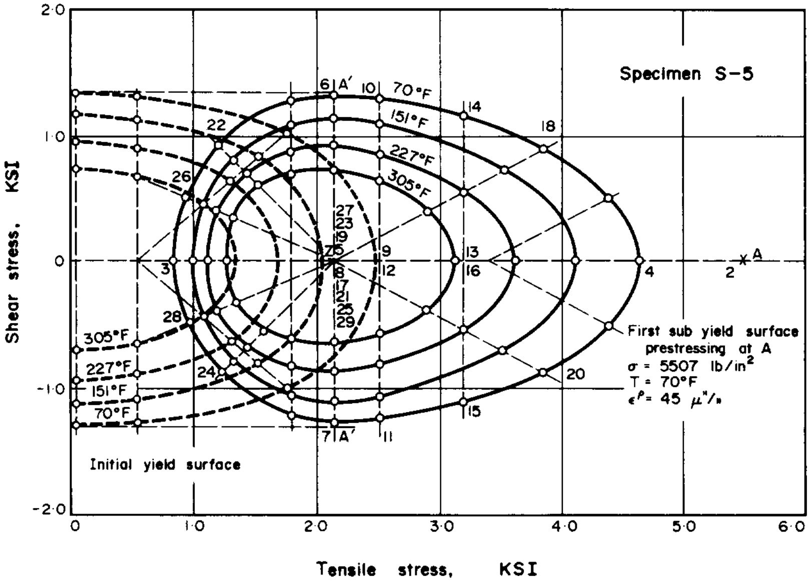

FIG. 3. First subsequent yield surface for specimen *S*-5.

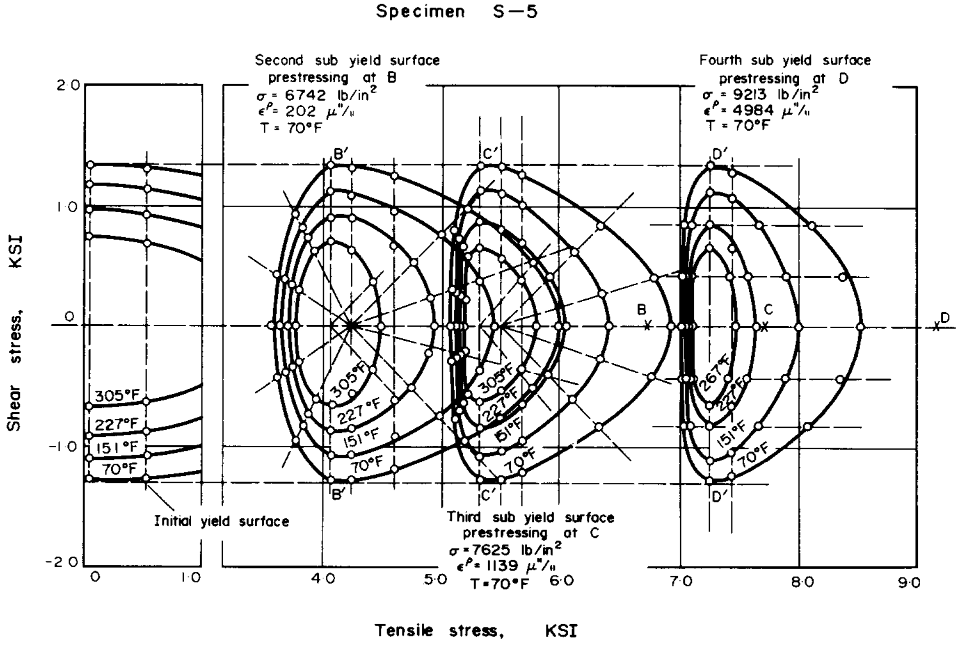

FIG. 4. Second, third and fourth subsequent yield surfaces for specimen *S*-5.

RESULTS

Five specimens have been tested; they are designated *S*-5, *S*-7, *S*-8, *S*-9, *S*-10. Superposition of the initial yield curves for these five specimens shows that the yield curves agree to within ± 3 per cent from specimen to specimen. The curves are ellipses intermediate between the Mises and Tresca ellipses. For each specimen these ellipses are the isothermals of a truncated elliptical cone in (σ, τ, T) space. The base of this cone is on the stress plane at 70°F and the apex (extrapolated) is on the temperature axis $T = 600$–650°F. Recall that 650°F is the annealing temperature of this material. The cone is truncated at 305°F corresponding to the lack of data above that temperature.

Figures 3 and 4 give the sequence of yield surfaces for specimen *S*-5. The first subsequent yield surface is due to prestressing to the point *A*; the second, third and fourth subsequent yield surfaces are due to prestressing to the points *B*, *C* and *D*, respectively. It is seen that there exists a remarkable lack of cross effect at all tested temperatures. This means that at each temperature the maximum positive and negative shear stresses reached by the yield curve after prestressing in tension are the same as the maximum shear stresses reached by it before prestressing. This phenomenon has been observed for torsional prestressing by Ivey [3], Naghdi *et al.* [4], Phillips [1] and Phillips *et al.* [2] but never before for prestressing in tension.

It is also remarkable that the maximum values of positive and negative shearing stress occur at the same value of tensile strength for all temperatures, as indicated by the lines $A'A'$, $B'B'$, $C'C'$ and $D'D'$. It is seen that the tensile stresses at which these maximum and minimum values of stress occur are independent of the temperature. We also observe that as prestressing progresses there is a decrease in the width of the yield curve measured in the current direction of prestressing.

It is seen that the hardening law illustrated by Fig. 1 is satisfied by the results shown for specimen *S*-5. Indeed, we can say that at each temperature the motion of the yield surface due to prestressing is due to the superposition of a rigid body motion in the direction of prestressing and of a deformation in the same direction. The rigid body motion is

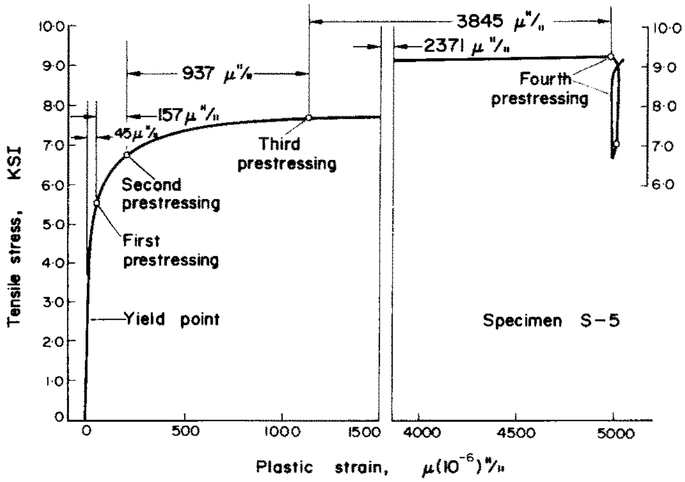

FIG. 5. Stress–plastic strain diagram for specimen *S*-5.

illustrated by the succession of lines, $A'A' \rightarrow B'B' \rightarrow C'C' \rightarrow D'D'$. The deformation is illustrated by the decrease in the width of the yield curves measured in the direction of prestressing.

We observe that the subsequent yield surfaces do not enclose the origin and that they never pass through the prestressing points.

For the fourth subsequent yield surface the maximum tested temperature is 267 and not 305°F as it is for all preceding subsequent yield surfaces. This is so because at 305°F there was no yield curve obtainable. At this temperature there is no region in stress space, which can be termed elastic, corresponding to prestressing point, *D*, and to the prestressing path used for specimen *S*-5. The size of the yield curve has become zero. Indeed, at 305°F, under the prestressing conditions prescribed, there is continuous creep (at the minimum rate of 7–8 μin./in. per min) at any point in stress space. We conclude that for every condition of prestressing there exists a maximum temperature beyond which the isothermal yield curve has zero size.

Figure 5 shows the stress–plastic strain diagram indicating the amount of plastic strain applied to specimen *S*-5. It is seen that while during first prestressing the amount of plastic strain is very small—45 μin./in.—the final plastic strain is approximately 1/2 per cent. Of particular interest is the fact that the yield surface does not enclose the origin even at the very minimal amount of first prestressing at $4{\cdot}5 \times 10^{-5}$ in./in.

Figure 6 refers also to specimen *S*-5. This figure gives the intersections of the yield surfaces with the plane passing through the prestressing path and parallel to the temperature axis. We observe that in the direction of prestressing, as prestressing increases, the intersections change from straight lines to increasingly concave curves. In addition, the fourth subsequent yield surface must be a closed one in the temperature direction with a highest temperature between 267 and 305°F.

Figures 7–10 give the sequences of subsequent yield surfaces for specimens *S*-7, *S*-8, *S*-9 and *S*-10, respectively. In all these tests the first subsequent yield surface is due to prestressing to the point *A*, while the second, and if obtained, third and fourth yield surfaces

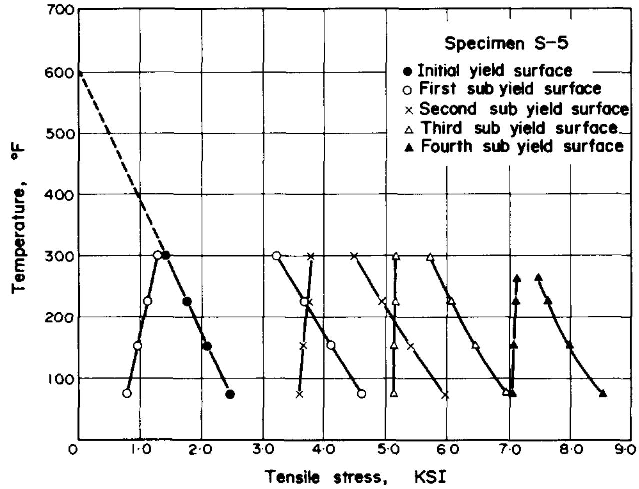

FIG. 6. Stress–temperature cross section of yield surfaces for specimen *S*-5.

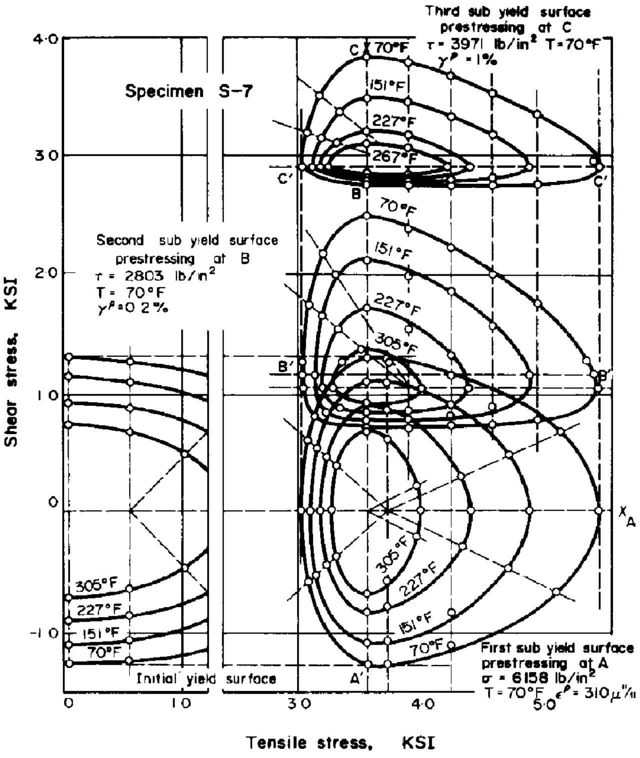

FIG. 7. Yield surfaces for specimen *S*-7.

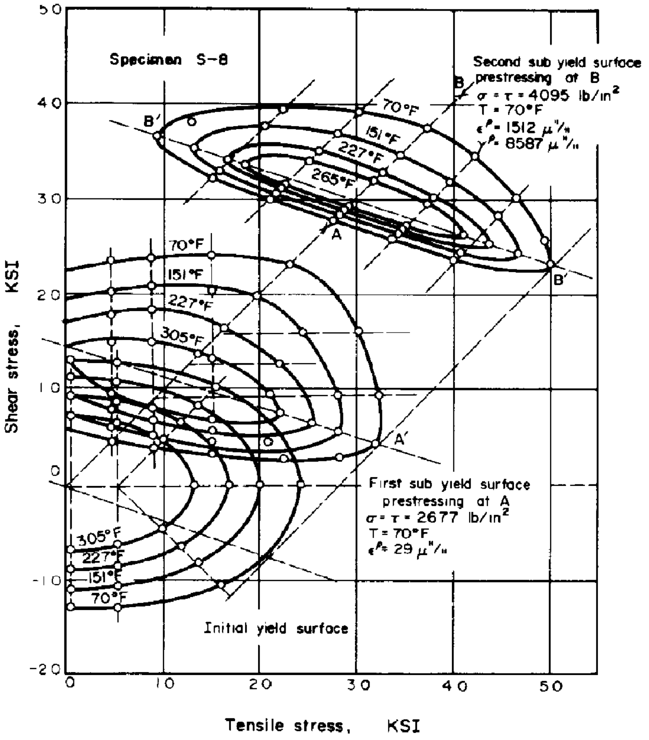

FIG. 8. Yield surfaces for specimen *S*-8.

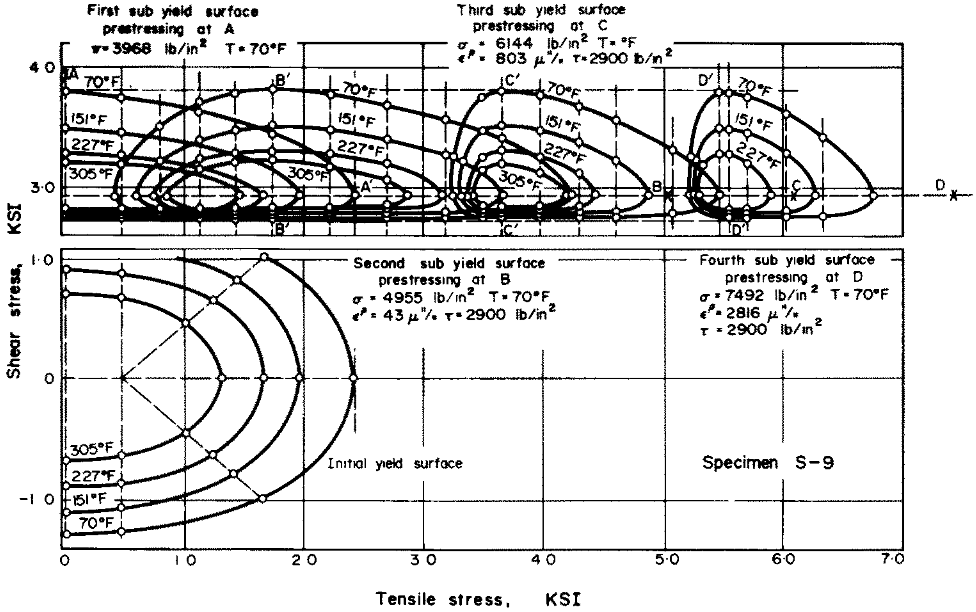

FIG. 9. Yield surfaces for specimen *S*-9.

are due to prestressing to points *B*, *C* and *D*, respectively. All conclusions obtained for specimen *S*-5 are also valid for specimens *S*-7, *S*-8, *S*-9 and *S*-10. In particular, the hardening law proposed by means of Fig. 1 is valid to a remarkable degree at every temperature.

Since normally a week was necessary to obtain a complete yield surface (at four temperature levels) and only then the subsequent yield surface was obtained, it is obvious

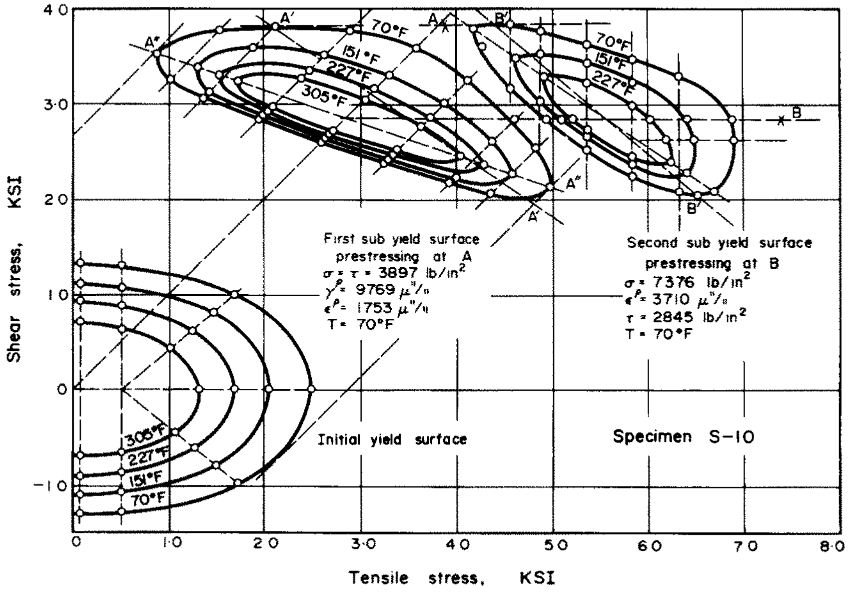

FIG. 10. Yield surfaces for specimen *S*-10.

that the above conclusions are not influenced by time-dependent phenomena. An additional verification of the stability of the yield surfaces is given by the following consideration. Each time a subsequent yield surface must be obtained it is necessary to cross the previous yield surface in order to establish a new prestress point. We consistently observed that the position of each previous yield surface, which was obtained 1–4 weeks previously, has not been altered by the passage of time.

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REFERENCES

[1] A. PHILLIPS, Yield Surfaces of Pure Aluminum at Elevated Temperatures, *Proceedings of the IUTAM Symposium on Thermoinelasticity*, edited by B. A. BOLEY, pp. 241–258. Springer (1970).

[2] A. PHILLIPS, C. S. LIU and J. W. JUSTUSSON, An experimental investigation of yield surfaces at elevated temperatures. *Acta Mech.* in press.

[3] P. M. NAGHDI, F. ESSENBURG and W. KOFF, An experimental study of initial and subsequent yield surfaces in plasticity. *J. appl. Mech.* **25**, 201–209 (1957).

[4] H. J. IVEY, Plastic stress–strain relations and yield surfaces for aluminum alloys. *J. mech. Engng Sci.* **3**, 15–31 (1961).

[5] S. B. BATDORF and B. BUDIANSKY, A Mathematical Theory of Plasticity Based on the Concept of Slip, NACA TN 1871 (1949).

[6] P. K. BERTSCH and W. N. FINDLEY, An Experimental Study of Subsequent Yield Surfaces—Corners, Normality, Bauschinger and Allied Effects, *Proceedings 4th U.S. National Congress of Applied Mechanics*, pp. 893–907. ASME (1962).

[7] G. I. TAYLOR and H. QUINNEY, The plastic distortion of metals. *Phil. Trans. R. Soc.* **A230**, 323–362 (1931).

[8] L. W. HU and J. F. BRATT, Effect of tensile plastic deformation on yield condition. *J. appl. Mech.* **25**, 411 (1958).

[9] J. MIASTKOWSKI and W. SZCZEPINSKI, An experimental study of yield surfaces of prestrained brass. *Int. J. Solids Struct.* **1**, 189–194 (1965).

[10] W. V. MAIR and H. L. D. PUGH, Effect of prestrain on yield surfaces in copper. *J. mech. Engng Sci.* **6**, 150–163 (1964).

[11] T. D. DUDDERAR and J. DUFFY, Neutron irradiation and the yield surfaces of copper. *J. appl. Mech.* **34**, 200–206 (1967).

[12] W. PRAGER, A new method of analyzing stress and strains in workhardening plastic solids. *J. appl. Mech.* **23**, 493–496 (1956).

[13] I. U. ISHLINSKII, General theory of plasticity with linear strain hardening. *Ukr. mat. Zh.* **6**, 314–324 (1954).

[14] P. G. HODGE, JR., Discussion of Ref. 12. *J. appl. Mech.* **24**, 482 (1957).

[15] H. ZIEGLER, A modification of Prager's hardening rule. *Q. appl. Math.* **17**, 55–65 (1959).

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Абстракт—Даются результаты зкспериментального исследования поверхностей текучести чистого алюминия, при повышенных температурах, в пределах 21,2–151,6°C. Используются некоторые пути предварительного напряжения в пространстве σ–τ. На основе этих результатов формулируется новый закон упрочнения. Кроме того, указывается, что поверхности текучести не переходят сквозь точек предварительного напряжения, и даже, при относительно малом предварительном напряжении, поверхности текучести не заключают начальной точки.